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State-Dependent Linear Utility Functions for Monetary Returns

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Abstract

We present a theory of expected utility with a state-dependent linear utility function for monetary returns, which includes results on first-order stochastic dominance, mean-preserving spread, increasing-concave linear utility profiles, and risk aversion. Applying the expected utility theory developed here, we analyze the contract a monopolist would offer in an insurance market that allows for partial loss coverage. We also define a utility function for monetary wealth that, in a certain sense, reconciles state-dependent constant average utility of money with loss aversion and the Friedman-Savage hypothesis. As an immediate consequence of such a utility function, we obtain a profile of state-dependent linear utility functions for monetary wealth, where states of nature correspond to intervals in which monetary wealth may occur. The intervals are mutually disjoint, and an interval corresponding to greater monetary wealth has a lower positive average utility of monetary wealth.

Keywords: Money, Utility function, State-dependent, Linear, First-order stochastic dominance, Mean-preserving spread, Risk aversion, Loss aversion, Friedman-savage hypothesis.

1 | Introduction

A common argument against a linear utility function for monetary returns is that an agent with such a utility function would have no incentive to insure himself against possible loss. However, this argument seems to collapse if the linear utility function for monetary returns is state-dependent. The probability of the gain or loss is spelled out as the probability of the state of nature (son) in which there is the gain or loss, with the constant marginal utility of monetary returns in the "worse" state being more than the constant marginal utility of money in the better state.

The seminal contribution of Kahneman and Tversky [1] noted the experimentally verified observation that agents tend to have a marginal utility of loss that is no less- if not higher than the marginal utility of gain, so

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that a typical utility function for monetary returns $u:\mathbb{R}\rightarrow \mathbb{R}$ may be of the form $u(x) = u+\max\{x,0\} + u-\min\{x,0\}$ with $u- \geq u+ > 0$. This phenomenon is known as "loss aversion". Thus, any utility function of this form can be represented by a pair of real numbers (u-, u+) where $u- \geq u+ > 0$. Allowance is made for the possibility of u- = u+.

The work of Friedman and Savage [2] clearly spells out that, beginning with an initial segment where the utility function for wealth is concave, the utility function alternates between convexity and concavity thereafter. This property of utility functions for wealth is the "Friedman-Savage" hypothesis. In [3], a utility function for wealth (expressed in terms of gains and losses) has been suggested as compatible with "loss aversion" as well as the "Friedman-Savage" hypothesis. However, the utility function in Lahiri [3] does not display constant average utility in any subinterval of its domain, and this is a problem for reasons that we now address.

Ramsey and de Finetti's dominant interpretation of probability in expected utility theory is that of Lahiri [4], [5], which provides brief discussions and intuitive motivations for such probabilities. The Ramsey-de Finetti subjective probability of an "event" or "state of nature" (say E) that an agent assesses is the price (say P) that the agent would be willing to pay for a simple bet that returns one unit of money if the state of nature 'E' occurs and nothing otherwise so that the expected monetary value of the simple bet to the agent is zero. Thus, if the average utility of money in state of nature E is a constant, say $\mu > 0$, then for one unit of money in state of nature E, the agent will be willing to forego μp units of utility and for ξ units of money in state of nature E the agent will be willing to forego $\mu p \xi$ units of utility, the latter being the utility the agent willingly forgoes for ξ simple bets of the type we have just discussed. ξ simple bets, each of which returns one unit of money if E occurs and nothing otherwise, are identical to a bet that returns ξ unit of money if E occurs and nothing otherwise. Thus, Ramsey-de Finetti subjective probability fits comfortably with "expected utility theory" based on constant average state-dependent utility. On the other hand, if the average utility in the state of nature E is "non-constant", then there exists ξ such that the average utility of ξ units of money is not equal to the average utility of $p\xi$ units of money. For a bet that returns ξ units of money in the state of nature E and nothing otherwise, the agent will be "willingly preceding" the utility of $p\xi$ units of money and not 'p' times the utility of ξ units of money, the latter being the expected utility of the bet to the agent. Hence, there seems to be a mismatch between Ramsey-de Finetti's subjective probability and expected utility theory based on such an interpretation of the state-dependent average utility of money as "non-constant". A comprehensive exposition of the early stages of analyzing decision-making under uncertainty with state-dependent preferences is available in [6]. However, the significance of state-dependent linear utility functions for money is that they fit comfortably with the expected utility concept based on Ramsey-de Finetti's probabilities. Thus, Ramsey-de Finetti probabilities and expected utilities are "perfectly economically consistent" with state-dependent linear utility functions for money.

In the next section of the paper, we motivate our discussion in the subsequent sections by considering a "toy model" of insurance against a risky loss. We apply expected state-dependent linear utility analysis in this model and show that insurance is possible under state-dependent "risk neutrality". The third section presents the formal framework for "expected utility theory with state-dependent linear utility functions for monetary returns". Using concepts introduced in this section, in subsequent sections, we introduce "first-order stochastic dominance", "mean-preserving spread", "increasing-concave linear utility profiles", and "risk aversion". Applying the expected utility theory developed here, we analyze the contract a monopolist would offer in an insurance market that allows for partial loss coverage.

Our final endeavour concerns amplifying ideas implicit in "increasing-concave linear utility profiles". We define a utility function for monetary wealth that, in a certain sense, reconciles state-dependent constant average utility of money with "loss aversion" and the "Friedman-Savage" hypothesis. As an immediate consequence of such a utility function, we obtain a profile of state-dependent linear utility functions for monetary wealth, where states of nature correspond to intervals in which monetary wealth may occur. The intervals are mutually disjoint, and an interval corresponding to greater monetary wealth has a lower positive average utility of monetary wealth.

We hope that this paper will provide an incremental impetus for the further development of decision analysis with linear utility functions for money.

2 | Motivation-Insuring Against Risky Loss

Consider a situation with two states of nature 1,2, where an agent with initial wealth $w > 0$ may face a loss of $L \in (0, w)$ units of money in the second son. Let $p \in (0,1)$ be the probability of loss. Suppose that his utility function for monetary returns in Son i is a function of the above form with (u_i^-, u_i^+) being the slopes for losses and gains respectively in Son 'i'.

There are two ways in which insurance can be introduced in this setting. First is a variation of the traditional textbook setting where we assume $u_2^- > u_1^-$. Even an individual unaffected by the loss would react to the news by leaning closer towards caution and hence a higher marginal utility of money than in the absence of such news. However small the difference in the marginal utilities may be. If one hears about frequent bicycle thefts in the neighbourhood that one lives in, then the same person is likely to be concerned more about the safety of their bicycle than they would be in the absence of such news, regardless of whether the person has been a victim of such theft or not. The difference gets more pronounced for an agent with a stake in the loss.

In the absence of an insurance policy, the expected utility of the agent is $-pu_2^-L$. An insurance policy that provides complete coverage is available for a premium π which, if actuarily fair, would satisfy $\pi = pL$.

The expected utility from buying this policy is $-[(1\text{-}p)\ u_1^- + pu_2^-]\pi = -p[(1\text{-}p)\ u_1^- + pu_2^-]L$. Since $u_2^- > u_1^- > 0$ and $p \in (0,1)$, $(1\text{-}p)\ u_1^- + pu_2^- < u_2^-$ and so $-p[(1\text{-}p)\ u_1^- + pu_2^-]L > -pu_2^-L$.

Actually, it would be more realistic to consider three sons: 1) where there is no loss, 2) where there is a loss and the agent has not bought the insurance policy, and 3) where there is a loss and the agent has bought the insurance policy, with $u_2^- > u_3^- > u_1^- > 0$, since having bought the insurance policy, the agent is somewhat more relaxed than they would have been had they not purchased it. Still, since recovering the insurance payment involves some transaction cost (e.g., paperwork, etc.), the agent's disutility from expenditure incurred on buying the premium could be expected to be higher than what it would have been had there been no loss.

A second way in which insurance can be introduced in this context, which may be more realistic, is to assume that the seller of the insurance policy has recourse to an investment opportunity, which for some $r > 0$, returns $1 + r$ units of money for every unit of money invested in the current period. In this case, we can weaken the restriction on the slopes of the utility functions and assume $u_2^- \geq u_1^-$, i.e., allow for $u_2^- = u_1^-$.

In this case, an insurance policy that provides complete coverage for a premium π, yields an expected return of $(1+r)\pi\text{-}pL$ to the seller of the insurance policy, which is non-negative if $\pi \geq \frac{pL}{1+r}$. Since $r > 0$, $\frac{pL}{1+r} < pL$, so that the seller of the policy can make a profit by selling it for a premium $\pi \in (\frac{pL}{1+r}, pL)$.

In this case, the expected utility from buying this policy for a premium of π is $-[(1\text{-}p)\ u_1^- + pu_2^-]\pi$ and $-[(1\text{-}p)\ u_1^- + pu_2^-]\pi > -pu_2^-L$, since $0 < (1\text{-}p)\ u_1^- + pu_2^- \leq u_2^-$ and $\pi < pL$.

Now let us consider an agent whose initial monetary wealth is $w > 0$ and an investible amount $I \in (0,w)$ can either be diversified equally between two risky investment opportunities or invested entirely in one investment opportunity, with each investment opportunity having a probability $p \in (0,1)$ of failing.

This is a situation where there are three states of nature denoted by 1, 2, 3, with (u_i^-, u_i^+) being the slopes for losses and gains respectively in son 'i'> 0. Son 1 is the situation where neither investment opportunity fails, Son 2 is where 50% of the invested amount is lost, and Son 3 is where the entire invested amount is lost.

Suppose $0 < u_1^- < u_2^- < u_3^-$.

Even if the agent were not an investor, the news of an investment opportunity crashing would likely increase expenditure disutility. Such disutility would increase if it heard the news of two investment opportunities crashing simultaneously. In the case of an investor, the effect of such news is likely to be more pronounced.

If the agent invests the entire amount I in exactly one investment opportunity, then his expected utility is – pu_3^-I.

If the agent spreads his investment opportunity equally between the two investment opportunities, then his expected utility is -2p(1-p) $u_2^-\frac{I}{2}$ - p2u_3^-I = -p[(1-p) u_2^- + pu_3^-]I.

Since u_3^- > (1-p) u_2^- + pu_3^-, we have -p[(1-p) u_2^- + pu_3^-]I > – pu_3^-I, and hence, there is always an incentive for "spreading risks".

3 | The Framework of Analysis

Let us set up the general analysis framework with linear utility functions for monetary returns. One may refer to Bonanno [7] for a more general analysis framework.

For some positive integer L ≥ 2, let {1, 2, …, L} denote the finite set of states of nature.

A (column) vector $x \in \mathbb{R}^L$ where for each j∈{1,…, L}, the jth coordinate of x denotes the monetary return in son j, is said to be a return vector.

A (column) vector $p \in \mathbb{R}^L_{++}$ satisfying $\sum_{j=1}^{L} p_j = 1$, such that for j∈{1,…,L}, pj > 0 is the probability of occurrence of son j, is a probability vector. Given x, $y \in \mathbb{R}^L$, let yTx denote $\sum_{j=1}^{L} y_j x_j$.

A portfolio of risky assets (briefly referred to as a pora) is a pair (x, p) where x is a return vector and p is a probability vector.

Given a pora (x, p) with X denoting the random monetary return for (x, p) and α∈ℝ, let {X = α} denote the event that the realized son yields a monetary return of α, {X ≤ α} denote the event that the realized son yields a monetary return less than or equal to α, {X ≥ α} denote the event that the realized son yields a monetary return greater than or equal to α, {X < α} denote the event that the realized son yields a monetary return less than α, {X > α} denote the event that the realized son yields a monetary return greater than α. Thus, for all α∈ℝ, Probability of {X ≤ α} = 1- Probability of {X > α}. The expected value of a pora (x, p) denoted E(x, p) is pTx = $\sum_{j=1}^{L} p_j\, x_j$.

A linear utility profile is a vector $u \in \mathbb{R}^L_{++}$ such that the trader's (Bernoulli) utility for monetary returns (gains or losses) in son j∈{1, …., L} is ujα for all real numbers α, with α denoting the monetary return in son j.

Given a linear utility profile α and a pora (x ,p) the expected utility of (x, p) for u, denoted by Eu(x, p) is $\sum_{j=1}^{L} p_j u_j\, x_j$.

Clearly Eu(x, p) = p1(u1x1 – u2x2) + (p1 + p2)(u2x2 – u3x3) + (p1 + p2 + p3)(u3x3 – u4x4) + … + (p1 + … + pL-1)(uL-1xL-1 – uLxL) + (p1 + p2 + … + pL)xL = $\sum_{j=1}^{L-1}(\sum_{k=1}^{j} p_k)\,(u_j\, x_j - u_{j+1}x_{j+1}) + (\sum_{k=1}^{L} p_k)$uLxL.

Given a linear utility profile α and a pora (x,p), the certainty equivalent of (x,p) for u, denoted by CE(u, x, p), is the scalar that satisfies [CE(u, x, p)]pTu = Eu(x, p).

Suppose that (x, p) is a pora satisfying xj < xj+1 for all j∈{1, …, L-1}.Then, for all k∈{1, …, L-1} and α, β∈(xk, xk+1), Probability of {X > α} = Probability of {X > xk} = Probability of {X > β} and Probability of {X ≤ α} = Probability of {X ≤ xk} = Probability of {X ≤ β}.

4 | First Order Stochastic Dominance

Given two poras (x, p) and (y, q), with X denoting the random monetary return for (x, p) and Y denoting the random monetary return for (y, q), we say that (x, p) stochastically dominates (y, q) in the first order, denoted by (x, p) >FSD (y,q) if for all $\alpha \in \mathbb{R}$, Probability of {X > α} ≥ Probability of {Y > α} and for some $\alpha \in \mathbb{R}$, Probability of {X > α} > Probability of {Y > α}.

The intuitive interpretation of (x, p) >FSD (y,q) is that given any monetary return α, the probability that the monetary return from (x, p) is greater than α is greater than or equal to the probability that the monetary return from (y, q) is at greater α, and for some monetary return the first probability is strictly greater than the second probability i.e., (x, p) is consistently more likely to yield better rewards better than (y, q).

We know that for a linear utility profile and a pora (x, p), Eu(x, p) = $\sum_{j=1}^{L-1}(\sum_{k=1}^{j} p_k)\,(u_j\,x_j - u_{j+1}x_{j+1}\,) + (\sum_{k=1}^{L} p_k)$uLxL.

Proposition 1. Let (x, p) and (x, q) be two poras satisfying xj < xj+1 for all j∈{1, …, L-1}. Then (x, p) >FSD (x, q) if and only if [Eu(x, p) > Eu(x, q) for all linear utility profile u satisfying ujxj < uj+1xj+1 for all j∈{1, …, L-1}].

Proof: Eu(x, p) – Eu(x, q) = [$\sum_{j=1}^{L-1}(\sum_{k=1}^{j} p_k)\,(u_j\,x_j - u_{j+1}x_{j+1}\,) + (\sum_{k=1}^{L} p_k)$uLxL] – [$\sum_{j=1}^{L-1}(\sum_{k=1}^{j} q_k)\,(u_j\,x_j - u_{j+1}x_{j+1}\,) + (\sum_{k=1}^{L} q_k)$uLxL] = $\sum_{j=1}^{L-1}(\sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k)\,(u_j\,x_j - u_{j+1}x_{j+1}\,) + (\sum_{k=1}^{L} p_k - \sum_{k=1}^{L} q_k)$uLxL = $\sum_{j=1}^{L-1}(\sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k)\,(u_j\,x_j - u_{j+1}x_{j+1})$, since $\sum_{k=1}^{L} p_k = 1 = \sum_{k=1}^{L} q_k$).

Suppose (x, p) >FSD (x, q). Then, $\sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k \leq 0$ for all j∈{1, …, L}, with strict inequality for at least one j∈{1, …, L-1}, since $\sum_{k=1}^{L} p_k = 1 = \sum_{k=1}^{L} q_k$.

If u is a linear utility profile satisfying ujxj<uj+1xj+1 for all j∈{1, …, L-1}, then $\sum_{j=1}^{L-1}(\sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k)\,(u_j\,x_j - u_{j+1}x_{j+1}\,) > 0$. Thus, Eu(x, p) – Eu(x, q) > 0, i.e., Eu(x, p) > Eu(x, q).

Now suppose that it is not the case that (x, p) >FSD (x, q). Thus, {j∈{1, …, L-1} | $\sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k > 0$} ≠ ϕ. Let $\eta = \min\{\sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k \mid \sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k > 0\}$.

Let u1 = 1. Having defined uj > 0, let uj+1 > 0 be such that uj+1xj+1 – ujxj = $\frac{2}{\eta}$ if $\sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k > 0$ and $\frac{1}{2L}$ > uj+1xj+1 – ujxj > 0, otherwise. This is possible since xj+1 > xj implies that it is not possible for both xj+1 and xj to be zero. Thus, Eu(x, p) – Eu(x, q) = $-\frac{2}{\eta}\sum_{h\in\{\{j\in\{1,\dots,L-1\}\mid \sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k > 0\}} \sum_{k=1}^{h} p_k - \sum_{k=1}^{h} q_k + \sum_{h\in\{\{j\in\{1,\dots,L-1\}\mid \sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k \leq 0\}}(\sum_{k=1}^{h} p_k - \sum_{k=1}^{h} q_k)\,(u_h x_h - u_{h+1}x_{h+1}) =$ $-\frac{2}{\eta}\sum_{h\in\{\{j\in\{1,\dots,L-1\}\mid \sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k > 0\}} \sum_{k=1}^{h} p_k - \sum_{k=1}^{h} q_k + \sum_{h\in\{\{j\in\{1,\dots,L-1\}\mid \sum_{k=1}^{j} p_k - \sum_{k=1}^{j} q_k \leq 0\}}(\sum_{k=1}^{h} q_k - \sum_{k=1}^{h} p_k)\,(u_{h+1}x_{h+1} - u_h x_h) \leq -2 + (L-1)\frac{1}{2L} \leq -2 + \frac{1}{2} = -\frac{3}{2} < 0$.

Thus, [Eu(x, p) > Eu(x, q) for all linear utility profile u satisfying ujxj < uj+1xj+1 for all j∈{1, …, L-1}] implies (x, p) >FSD (x, q). Q.E.D.

5 | Mean-Preserving Spread and Increasing-Concave Linear Utility Profiles

For this section, assume L ≥ 3. Given a return vector x satisfying xj < xj+1 for all j∈{1, …, L-1}, a linear utility profile u is said to be increasing-concave with respect to x, if for all j∈{1, …, L-1}, ujxj < uj+1xj+1 and for all i, j, k∈{1, 2, …, L} with i < j < k, ujxj > (1-δ)uixi + δukxk where δ∈(0,1) satisfies xj = (1-δ)xi + δxk. Clearly, $\delta = \frac{x_j - x_i}{x_k - x_i}$ and 0 < xj – xi < xk – xi.

Given a return vector x satisfying xj < xj+1 for all j∈{1, …, L-1}, pora (x, q) is said to be obtained by a mean-preserving spread from pora (x, p), denoted (x, p) →MSP (x, q), if E(x, p) = E(x, q) and there exists i, j, k∈{1, 2, …, L} satisfying i < j < k such that qi > pi, qj < pj, qk > pk and ph = qh for h∈{1, 2, …, L}\{i,j,k}.

[E(x, p) = E(x, q) and there exists i, j, k∈{1, 2, …, L} satisfying i < j < k such that qi > pi, qj < pj, qk > pk and ph = qh for h∈{1, 2, …, L}\{i,j,k}] if and only if [there exists i, j, k∈{1, 2, …, L} satisfying i < j < k such that qi > pi, qj < pj, qk > pk, ph = qh for h∈{1, 2, …, L}\{i,j,k} and (pj-qj)xj = (qi- pi)xi + (qk – pk)xk], [there exists i, j, k∈{1, 2, …, L} satisfying i < j < k such that qi > pi, qj < pj, qk > pk and ph = qh for h∈{1, 2, …, L}\{i,j,k}and (pj-qj)xj = (qi- pi)xi + (qk – pk)xk] is equivalent to [there exists i, j], k∈{1, 2, …, L} satisfying i < j < k such that qi > pi, qj < pj, qk > pk, ph = qh for h∈{1, 2, …, L}\{i,j,k} and xj = $\frac{q_i-p_i}{p_j-q_j}$ xi + $\frac{q_k-p_k}{p_j-q_j}$xk].

Thus, (x, p) →MSP (x, q) if and only if [there exists i, j, k∈{1, 2, …, L} satisfying i < j < k such that qi > pi, qj < pj, qk > pk, ph = qh for h∈{1, 2, …, L}\{i,j,k} and xj = $\frac{q_i-p_i}{p_j-q_j}$ xi + $\frac{q_k-p_k}{p_j-q_j}$xk].

Proposition 2. Let (x, p) and (x, q) be two poras satisfying xj < xj+1 for all j∈{1, …, L-1}.

I. If (x, p) →MSP (x, q), then [Eu(x, p) > Eu(x, q) for all linear utility profile u which is increasing-concave with respect to x].

II. If L = 3, p2 ≠ q2 and [Eu(x, p) > Eu(x, q) for all linear utility profile u which is increasing-concave with respect to x] then (x, p) →MSP (x, q).

Proof: Suppose (x, p) →MSP (x, q) and let u be an increasing-concave linear utility profile with respect to x.

Hence, there exists i, j, k∈{1, 2, …, L} satisfying i < j < k such that qi > pi, qj < pj, qk > pk and ph = qh for h∈{1, 2, …, L}\{i,j,k}and (pj-qj)xj = (qi- pi)xi + (qk – pk)xk] is equivalent to [there exists i, j, k∈{1, 2, …, L} satisfying i < j < k such that qi > pi, qj < pj, qk > pk, ph = qh for h∈{1, 2, …, L}\{i,j,k} and xj = $\frac{q_i-p_i}{p_j-q_j}$ xi + $\frac{q_k-p_k}{p_j-q_j}$xk.

However, xj = (1-δ)xi + δxk where δ = $\frac{x_j-x_i}{x_k-x_i}$∈(0, 1). Further, pi + pj + pk = qi + qj + qk implies pj – qj = (qi – pi) + (qk – pk). Thus, $\frac{q_i-p_i}{p_j-q_j}$ + $\frac{q_k-p_k}{p_j-q_j}$ = 1, with $\frac{q_i-p_i}{p_j-q_j}$ > 0 and $\frac{q_k-p_k}{p_j-q_j}$ > 0. Hence, $\frac{q_k-p_k}{p_j-q_j}$ = δ and $\frac{q_i-p_i}{p_j-q_j}$ = 1- δ.

Since u is increasing-concave ujxj > (1-δ)uixi + δukxk. Thus, (pj – qj)ujxj > (qi – pi)uixi + (qk – pk)ukxk, i.e., piuixi + pjujxj + pkukxk > qiuixi + qjujxj + qkukxk. Since, ph = qh for h∈{1, 2, …, L}\{i,j,k}, we get Eu(x, p) > Eu(x, q).

Now suppose L = 3 and x1 < x2 < x3 and p2 ≠ q2. We have p1 + p2 + p3 = q1 + q2 + q3 = 1.

Suppose, E(x, p) = E(x, q). Thus, p1x1 + p2x2 + p3x3 = q1x1 + q2x2 + q3x3.

Suppose, Eu(x, p) > Eu(x, q) for all linear utility profiles satisfying u1x1 < u2x2 < u3x3 and u2x2 > (1-δ)u1x1 + δu2x3, where x2 = (1-δ)x1 + δx3.

Since p2 – q2 ≠ 0, (p2-q2)x2 = (q1 – p1)x1 + (q3 – p3)x3 implies x2 = $\frac{q_1-p_1}{p_2-q_2}$x1 + $\frac{q_3-p_3}{p_2-q_2}$x3 = $\frac{q_1-p_1}{p_2-q_2}$x1 + $\frac{(1-q_1-q_2)-(1-p_1-p_2)}{p_2-q_2}$x3 = $\frac{q_1-p_1}{p_2-q_2}$x1 + $\frac{(p_2-q_2)-(q_1-p_1)}{p_2-q_2}$x3 = x3 - $\frac{q_1-p_1}{p_2-q_2}$(x3 – x1), i.e., x2 = x3 - $\frac{q_1-p_1}{p_2-q_2}$(x3 – x1). x2 < x3 and x3 > x1 implies $\frac{q_1-p_1}{p_2-q_2}$ > 0.

Similarly, x2 = $\frac{q_1-p_1}{p_2-q_2}$x1 + $\frac{q_3-p_3}{p_2-q_2}$x3 = $\frac{(1-q_2-q_3)-(1-p_2-p_3)}{p_2-q_2}$x1 + $\frac{q_3-p_3}{p_2-q_2}$x3 = $\frac{(p_2-q_2)-(q_3-p_3)}{p_2-q_2}$x1 + $\frac{q_3-p_3}{p_2-q_2}$x3 = x1 + $\frac{q_3-p_3}{p_2-q_2}$(x3 – x1). x2 > x1 and x3 > x1 implies $\frac{q_3-p_3}{p_2-q_2}$ > 0.

Thus, x2 = $\frac{q_1-p_1}{p_2-q_2}$x1 + $\frac{q_3-p_3}{p_2-q_2}$x3, x2 = (1-δ)x1 + δx3, δ > 0, 1- δ > 0, $\frac{q_3-p_3}{p_2-q_2}$ > 0, $\frac{q_1-p_1}{p_2-q_2}$ > 0 and x1 < x2 < x3 implies δ = $\frac{q_3-p_3}{p_2-q_2}$ and 1- δ = $\frac{q_1-p_1}{p_2-q_2}$. Thus, u2x2 >$\frac{q_1-p_1}{p_2-q_2}$u1x1 + $\frac{q_3-p_3}{p_2-q_2}$u3x3.

If p2 < q2, then (p2-q2)u2x2 < (q1- p1)u1x1 + (q3 – p3)u3x3 and thus, Eu(x, p) = p1u1x1 + p2u2x2 + p3u3x3 < q1u1x1 + q2u2x2 + q3u3x3 = Eu(x, q), leading to a contradiction.

Thus, it must be the case that p2 > q2. Hence, $\frac{q_1-p_1}{p_2-q_2}$ > 0 implies q1 > p1 and $\frac{q_3-p_3}{p_2-q_2}$ > 0 implies q3 > p3. Thus, we have (x, p) →MSP (x, q). Q.E.D.

Note: The proof of part II in *Proposition 2* can likely be extended to L > 3.

6 | Risk Aversion

Given a pora (x, p), an agent with a linear utility profile u is said to be:

I. Risk averse relative to (x,p) if E(x, p) > CE(u, x, p).

II. Risk neutral relative to (x,p) if E(x, p) = CE(u, x, p).

III. Risk-loving/seeking relative to (x, p) if E(x, p) < CE(u, x, p).

Example 1. Let L = 2, u1 = 1 and u2 = 2.

Let (x, p) = ((2,0), ($\frac{1}{2}$, $\frac{1}{2}$)). Thus, E(x, p) = 1.

In this case, Eu(x,p) = 1 and pTu = $\frac{3}{2}$, so that CE(u, x, p) = $\frac{2}{3}$ < 1 = E(x,p). Thus, the agent is risk averse relative to ((2,0), ($\frac{1}{2}$, $\frac{1}{2}$)).

Now let (x, p) = ((0,2), ($\frac{1}{2}$, $\frac{1}{2}$)). Once again, E(x, p) = 1.

Now, Eu(x,p) = 2 and since pTu = $\frac{3}{2}$, we have CE(u, x, p) = $\frac{4}{3}$ > 1 = E(x, p). Thus, the same agent is risk-loving/seeking relative to ((0,2), ($\frac{1}{2}$, $\frac{1}{2}$)).

Now suppose (x, p) = ((1,1), ($\frac{1}{2}$, $\frac{1}{2}$)). Once again, E(x, p) = 1.

Now, Eu(x,p) = $\frac{3}{2}$ and since pTu = $\frac{3}{2}$, we have CE(u, x, p) = 1 = E(x, p). Thus, the same agent is now risk neutral relative to ((1,1), ($\frac{1}{2}$, $\frac{1}{2}$)).

Given a pora (x,p) and a linear utility profile u, the risk premium relative to (x,p) denoted R(u, x, p) = E(x, p) – CE(u, x, p). Thus, $\sum_{j=1}^{L} p_j u_j\,(E(x,p) - R(u,x,p)) = \sum_{j=1}^{L} p_j u_j\, CE(u,x,p)$ = Eu(x, p).

If the agent is:

I. Risk averse relative to (x, p), then R(u, x, p) > 0.

II. Risk-loving/seeking relative to (x, p), then R(u, x, p) < 0.

III. Risk neutral relative to (x, p), then R(u, x, p) = 0.

Given two linear utility profiles, u, v, and two poras (x, p), (y, q), we say that u relative to (x, p) is more risk averse than v relative to (y, q) if R(u, x, p) > R(v, y, q).

7 | Insurance Contracts with the Possibility of Partial Coverage

As before, consider a situation with two states of nature, 1, 2, where an agent with initial wealth w > 0 may face a loss of L∈(0, w) units of money in the second son. Let p∈(0,1) be the probability of loss. Suppose the agent's linear utility profile is (u1, u2) with 0 < u1 < u2.

The expected value of the "risk" is -pL.

In the absence of an insurance policy, the agent's expected utility is – pu2L.

If CE1 is the certainty equivalent without an insurance policy, then [(1-p)u1 + pu2]CE1 = -pu2L. Thus, CE1 $= \frac{-pu_2L}{(1-p)u_1+pu_2}$ = -pL$\frac{u_2}{(1-p)u_1+pu_2}$.

An insurance policy with a deductible d∈[0, L) (i.e., in case of loss, the insurer pays L-d to the agent) is available for a premium π.

Hence, the expected profit of the insurer is π - p(L-d). For the insurer to voluntarily sell the insurance, it must be "profitable", i.e., π - p(L-d) ≥ 0.

Thus, profitability is equivalent to the condition - pL ≥ - (π + pd).

The expected value of this policy to the agent is – (π + pd).

The expected utility of the agent from buying this policy is -(1-p)u1π - pu2(π + d) = -[(1-p)u1 + pu2]π - pu2d.

For the agent to voluntarily buy the insurance, it must be the case that -[(1-p)u1 + pu2]π - pu2d ≥ – pu2L, i.e., -π - $\frac{u_2}{(1-p)u_1+pu_2}$ pd ≥ CE1, -π - $\frac{u_2}{(1-p)u_1+pu_2}$ pd = – (π + pd) + pd[1- $\frac{u_2}{(1-p)u_1+pu_2}$]. Thus, the agent will voluntarily by the insurance policy if and only if - (π + pd) + pd[1- $\frac{u_2}{(1-p)u_1+pu_2}$] ≥ CE1.

A profit-maximizing insurer will choose an insurance contract, i.e., a pair (π, d) that

$$\text{Maximizes } \pi - p(L-d),$$

$$\text{s.t. } \pi - p(L-d) \geq 0, -[(1-p)u1 + pu2]\pi - pu2d \geq - pu2L \text{ and } d \in [0, L).$$

The above problem is equivalent to choosing a pair (π, d) that

$$\text{Maximizes } \pi + pd,$$

$$\text{s.t. } \pi + pd \geq pL, [(1-p)u1 + pu2]\pi + pu2d \leq pu2L \text{ and } d \in [0, L).$$

It is easy to see that at an optimal solution, [(1-p)u1 + pu2]π + pu2d = pu2L. Thus, π = $\frac{pu_2(L-d)}{(1-p)u_1+pu_2}$. Thus, π + pd = p[$\frac{u_2(L-d)}{(1-p)u_1+pu_2}$ + d] = pd[1 - $\frac{u_2}{(1-p)u_1+pu_2}$] + $\frac{pu_2L}{(1-p)u_1+pu_2}$.

Since u2 > u1, we have $\frac{u_2}{(1-p)u_1+pu_2}$ > 1 and hence 1 - $\frac{u_2}{(1-p)u_1+pu_2}$ < 0.

Thus, pd[1 - $\frac{u_2}{(1-p)u_1+pu_2}$] + $\frac{pu_2L}{(1-p)u_1+pu_2}$ is maximized at d = 0, thereby implying π = $\frac{pu_2L}{(1-p)u_1+pu_2}$.

Since $\frac{pu_2L}{(1-p)u_1+pu_2}$ = ($\frac{u_2}{(1-p)u_1+pu_2}$)pL and $\frac{u_2}{(1-p)u_1+pu_2}$ > 1, we have π > pL. Since d = 0, π + pd > pL.

Hence, the optimal contract is the pair ($\frac{pu_2L}{(1-p)u_1+pu_2}$, 0), with the "expected profit of the insurer" being $\frac{pu_2L}{(1-p)u_1+pu_2}$ – pL = pL($\frac{u_2-(1-p)u_1-pu_2}{(1-p)u_1+pu_2}$) = $\frac{p(1-p)(u_2-u_1)L}{(1-p)u_1+pu_2}$ > 0.

Note: π = $\frac{pu_2(L-d)}{(1-p)u_1+pu_2}$ implies -π - $\frac{u_2}{(1-p)u_1+pu_2}$ pd = - $\frac{pu_2L}{(1-p)u_1+pu_2}$ = CE1.

We know that -π - $\frac{u_2}{(1-p)u_1+pu_2}$ pd = - (π + pd) + pd[1- $\frac{u_2}{(1-p)u_1+pu_2}$]. Thus, at an optimal solution - (π + pd) + pd[1- $\frac{u_2}{(1-p)u_1+pu_2}$] = CE1.

Strict Profitability is equivalent to the condition - pL > - (π + pd), which now reduces to -pL + pd[1- $\frac{u_2}{(1-p)u_1+pu_2}$] > CE1 = - $\frac{pu_2L}{(1-p)u_1+pu_2}$.

Thus, strict profitability is equivalent to -pL[1- $\frac{u_2}{(1-p)u_1+pu_2}$] + pd[1-- $\frac{u_2}{(1-p)u_1+pu_2}$] > 0, i.e. p(d-L)[1- $\frac{u_2}{(1-p)u_1+pu_2}$] ≥ 0.

Since d∈[0, L), this is possible if and only if 1- $\frac{u_2}{(1-p)u_1+pu_2}$ < 0, i.e., 1 < $\frac{u_2}{(1-p)u_1+pu_2}$.

Multiplying throughout by pL, which is strictly positive, we get 1 < $\frac{u_2}{(1-p)u_1+pu_2}$ if and only if pL < $\frac{u_2}{(1-p)u_1+pu_2}$pL, the latter being equivalent to - $\frac{u_2}{(1-p)u_1+pu_2}$pL < -pL.

Since CE1 = - $\frac{u_2}{(1-p)u_1+pu_2}$pL and -pL are the expected value of the risk. Thus, Strict Profitability is equivalent to the requirement that the agent is risk averse relative to ((-L, 0), (p, 1-p)).

Let us now consider the somewhat more realistic situation with three sons: 1) where there is no loss, 2) where there is a loss and the agent has not bought the insurance policy, and 3) where there is a loss and the agent has bought the insurance policy, with u2 > u3 > u1 > 0.

Then, the expected utility of the agent from buying this policy is -(1-p)u1π - pu3(π + d) = -[(1-p)u1 + pu3]π - pu3d.

Since u2 > u3, -(1-p)u1π - pu3(π + d) > -(1-p)u1π - pu2(π + d).

A profit-maximizing insurer will choose an insurance contract, i.e., a pair (π, d) that

$$\text{Maximizes } \pi - p(L-d),$$

$$\text{s.t. } \pi - p(L-d) \geq 0, -[(1-p)u1 + pu3]\pi - pu3d \geq -pu2L \text{ and } d \in [0, L].$$

The above problem is equivalent to choosing a pair (π, d) that

$$\text{Maximizes } \pi + pd,$$

$$\text{s.t. } \pi + pd \geq pL, [(1-p)u1 + pu3]\pi + pu3d \leq pu2L \text{ and } d \in [0, L].$$

It is easy to see that at an optimal solution, [(1-p)u1 + pu3]π + pu3d = pu2L. Thus, π = $\frac{p(u_2L-u_3d)}{(1-p)u_1+pu_3}$. Thus, π + pd = p[$\frac{u_2L-u_3d}{(1-p)u_1+pu_3}$ + d] = pd[1 - $\frac{u_3}{(1-p)u_1+pu_3}$] + $\frac{pu_2L}{(1-p)u_1+pu_3}$.

Since u3 > u1, we have $\frac{u_3}{(1-p)u_1+pu_3}$ > 1 and hence 1 - $\frac{u_3}{(1-p)u_1+pu_3}$ < 0. Thus, pd[1 - $\frac{u_3}{(1-p)u_1+pu_3}$] + $\frac{pu_2L}{(1-p)u_1+pu_3}$ is maximized at d = 0, thereby implying π = $\frac{pu_2L}{(1-p)u_1+pu_3}$ > $\frac{pu_2L}{(1-p)u_1+pu_2}$, since u3 < u2.

Since $\frac{pu_2L}{(1-p)u_1+pu_3}$ = ($\frac{u_2}{(1-p)u_1+pu_3}$)pL and $\frac{u_2}{(1-p)u_1+pu_3}$ > $\frac{u_2}{(1-p)u_1+pu_2}$ > 1, we have π > $\frac{pu_2L}{(1-p)u_1+pu_2}$ pL. Since d = 0, π + pd > $\frac{pu_2L}{(1-p)u_1+pu_2}$ > pL.

Hence, the optimal contract is the pair ($\frac{pu_2L}{(1-p)u_1+pu_3}$, 0), with the expected profit of the insurer being $\frac{pu_2L}{(1-p)u_1+pu_3}$ – pL = pL($\frac{u_2-(1-p)u_1-pu_3}{(1-p)u_1+pu_2}$) > $\frac{p(1-p)(u_2-u_1)L}{(1-p)u_1+pu_2}$ > 0. Thus, the expected profit of the insurer is higher in this more realistic situation than in the earlier situation.

8 | Almost Linear Utility Function for Monetary Wealth

For a non-negative integer n, let <u2j | j∈{0, …, n}> be a finite sequence of positive real numbers satisfying u2(j+1) < u2j if n > 0.

If n > 0, let N = 2n and let <xj | j∈{0, 1, .., N}> be a finite sequence of non-negative real numbers satisfying:

I. x0 = 0, (ii) xj+1 > xj, for all j∈{0, .., N-1}.

II. For all j∈{0, …, n-1}, u2jx2j+1 ≤ u2j+2x2j+2.

Let u: $\mathbb{R}_+ \to \mathbb{R}$ be the function defined thus:

I. If n = 0, then let u(x) = u0x for all $x \in \mathbb{R}_+$.

II. If n > 1, then for j∈{0, …, n-1}, let u(x) = u2jx for all x∈[x2j, x2j+1], u(x) = u2jx2j+1 + (x-x2j+1)$\frac{u_{2j+2}x_{2j+2} - u_{2j}x_{2j+1}}{x_{2j+2} - x_{2j+1}}$ for all x∈[x2j+1, x2j+2], and u(x) = u2nx for all x ≥ xN.

We will refer to such a function u as an Almost Linear Utility (ALU) function for monetary wealth.

Note: Compatibility with an ALU function with loss aversion requires that n > 0 and the initial wealth is at x2j-1 for some j∈{0, …, n}. If we require compatibility with the "Friedman-Savage" hypothesis, we require n > 1.

9 | State-Dependent Linear Utility Functions for Monetary Wealth

In the context of the ALU function defined in the previous section:

I. If n = 0, then the average utility of monetary wealth is the positive constant u0 for all $x \in \mathbb{R}_+$.

II. If n > 0, then the average utility of monetary wealth is a positive constant u2j in the interval [x2j, x2j+1] for all j∈{0, …, n-1} and is the positive constant u2n for x ≥ xN.

If n > 0, then for j∈{0, …, n- 1}, let Ej = [x2j, x2j+1] and let En = [x2n, + ∞).

Let X = $\bigcup_{j=0}^{n} E_j$be the sample space. {E1, …, En} can be considered a collection of n mutually exclusive events (or states of nature), each of which the average utility of money is a constant. For j∈{0, …, n- 1}, let u(.|Ej): Ej → ℝ be the function such that for all x∈Ej, u(x|Ej) = ujx. The L-tuple (u(.|E0), …, u(.|En)) is a profile of state-dependent linear utility functions for monetary wealth.

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Conflicts of Interest

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